Gravitational orbits

- circular
- perturbation of circular orbit: example of "small oscillations"

Let's take as our 2-body problem potential the gravitational potential energy of two point masses:

$$V(r) = -G\frac{M_1 M_2}{r} = -\frac{A}{r}$$

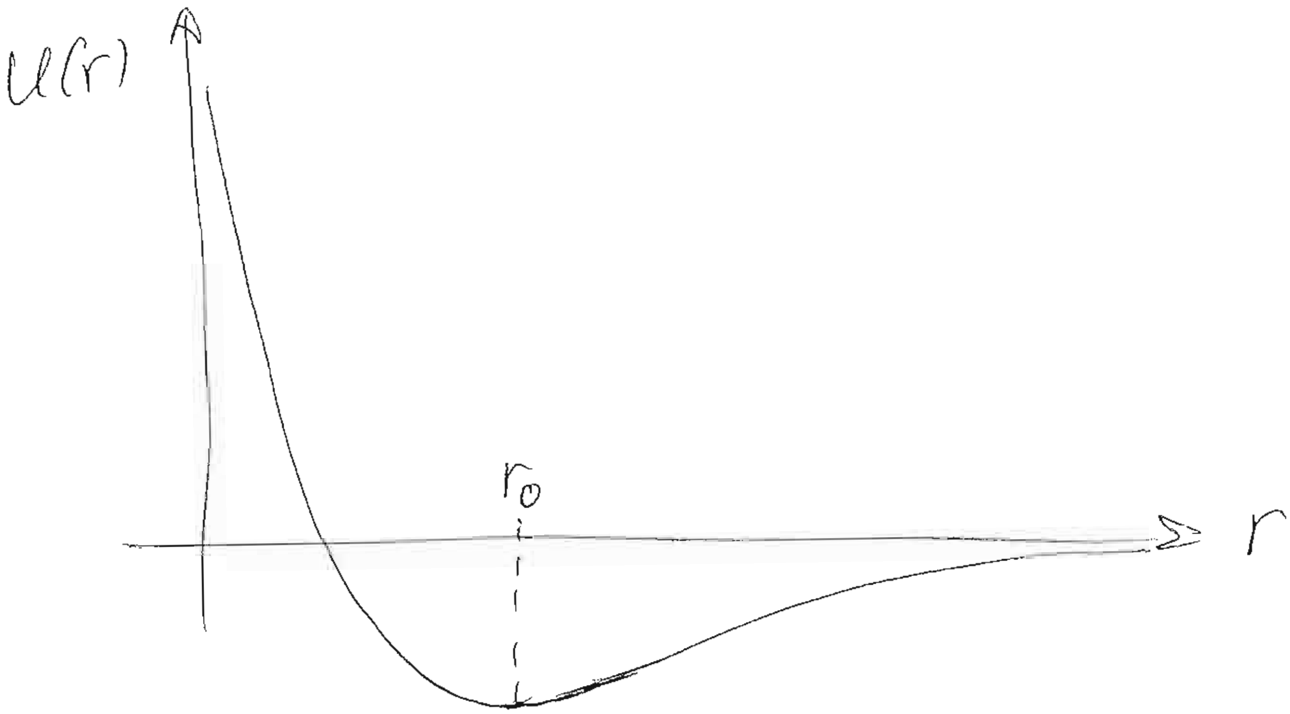
Using the identity

$$\frac{L_z^2}{\mu r^3} = -\frac{d}{dr}\left(\frac{1}{2}\frac{L_z^2}{\mu r^2}\right) = -\frac{d}{dr}\left(\frac{B}{r^2}\right),$$

we can write the equation of motion for the relative distance in terms of a single "effective potential":

$$\mu \ddot{r} = -\frac{dU}{dr}$$

$$U(r) = \frac{B}{r^2} - \frac{A}{r} \qquad \left(\begin{array}{l} A = G M_1 M_2 \\ B = \dfrac{L_z^2}{2\mu} \end{array} \right)$$



The attractive "A" term, due to gravity, dominates at large r while the repulsive "B" term, called the "centrifugal barrier", dominates at small r. When $L_z = 0$ there is no barrier for the two masses to simply fall into each other.

The radius r_0, defined by

$$\left.\frac{dU}{dr}\right|_{r_0} = 0,$$

is special. When the initial conditions are

$$r(0) = r_0$$
$$\dot{r}(0) = 0,$$

then the equation of motion

$$\mu\ddot{r}(0) = -\left.\frac{dU}{dr}\right|_{r(0)} = 0$$

implies $r(t)$ will stay constant.

This corresponds to a circular orbit. For this to happen the separation must be at the minimum r_0 of the effective potential:

$$0 = \left.\frac{dU}{dr}\right|_{r_0} = -\frac{2B}{r_0^3} + \frac{A}{r_0^2}$$

$$\Rightarrow \quad r_0 = \frac{2B}{A} = \frac{L_z^2/\mu}{GM_1M_2}$$

And since both L_z and $r = r_0$ are constant for circular orbits,

$$\dot{\theta} = \frac{L_z}{\mu r_0^2} = \omega_0 = \text{constant}$$

$$\Rightarrow \quad r_0 = \frac{\mu}{GM_1M_2}\left(\frac{L_z}{\mu}\right)^2 = \frac{\mu}{GM_1M_2}\left(r_0^2\omega_0\right)^2$$

$$\Rightarrow \quad G(M_1+M_2) = \omega_0^2 \, r_0^3$$

This is Kepler's famous law of orbital motion, for the special case of circular orbits.

Let's study what happens when the circular orbit is slightly <u>perturbed</u>. We'll assume the perturbation δr in the separation is so slight that we can approximate the function $U(r)$ near r_0 by a parabola:

$$U(r) \cong U(r_0) + \underbrace{U'(r_0)}_{0}(r-r_0) + \tfrac{1}{2}U''(r_0)(r-r_0)^2$$

$$= U_0 + \tfrac{1}{2} K \, \delta r^2$$

$$K = \frac{6B}{r_0^4} - \frac{2A}{r_0^3} = \frac{3}{r_0^3}A - \frac{2A}{r_0^3} = \frac{A}{r_0^3}$$

The equation of motion for the perturbation is

$$\mu \ddot{\delta r} = -\frac{dU}{dr} = -K\,\delta r\,.$$

So δr behaves as a simple harmonic oscillator with frequency

$$\omega = \sqrt{\frac{K}{\mu}}$$

What is the relationship between the two angular frequencies?

ω_0 : frequency of θ (orbit completion)

ω : frequency of δr (radial oscillations)

$$\omega^2 = \frac{K}{\mu} = \frac{A}{\mu r_0^3} = \frac{G(M_1+M_2)}{r_0^3}$$

$$\omega_0^2 = \frac{G(M_1+M_2)}{r_0^3} \quad \text{(Kepler)}$$

They are the same! In other words, the time taken to make one radial oscillation is precisely the time to complete one orbit.

Actually, we only know that L_z is constant, so $\dot{\theta} = \frac{L_z}{\mu r^2}$ is no longer constant when r varies with time. But since $\delta r \ll r_0$, r stays nearly constant and the constancy of $\dot{\theta}$ is violated by at most $O(\delta r / r_0)$.

The apparent coincidence that $\omega = \omega_0$ is consistent with the observation that there are elliptic orbits that do not <u>precess</u>: